

Approximation and Computation of Arbitrage in Frictional Foreign Exchange Market

(Extended Abstract)

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Abstract

In this paper we study the computational complexity of arbitrage in a frictional foreign exchange market with bid-ask spreads, bound and integrality constraints. We show that the problem of detecting the existence of arbitrage is *NP*-complete in the general case and, moreover, for some fixed $\epsilon > 0$, approximating the optimal version of the problem within a factor of n^ϵ is *NP*-hard where n is the number of foreign currencies. On the other hand, we show that the optimal problem can be solved in polynomial time for two special cases of the constant number of currencies or a star-shaped exchange graph.

1 Introduction

Arbitrage is, arguably, the most important concept in finance. It is commonly assumed that state variables of financial instruments will disallow the existence of investment strategies with riskless profit, commonly referred to as

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arbitrage opportunities. Active investment agents will exploit any arbitrage opportunity in a financial market and thus will deplete it as soon as it arises. Experimental results in foreign exchange market, however, showed that this theoretical doctrine might not always be true in reality. Examination of data from ten markets over a twelve day period by Mavrides revealed that a significant arbitrage opportunity exists [13]. Some opportunities were observed to be persistent for a long time. The problem become worse at forward and futures markets coupled with covered interest rates, as observed by Abeysekera and Turtle [1], and Clinton [3]. Different reasons were proposed and examined. In this work we are interested in this problem from a computational complexity point of view.

Recently, there has been a growing research effort in re-examination of concepts in humanity and social sciences via computational complexity approach: cooperative games [5], organizational structure [6], general equilibrium [7]. An arbitrage condition in a frictional security market has also been shown NP-hard under some realistic market conditions [4]. While NP-hardness usually implies difficulties in finding a solution, the NP-hardness result for arbitrage seems to be against the above well accepted doctrine of economics and finance. Even assuming that an NP-hardness result here indeed implies the hardness of finding all the arbitrage opportunities, it is still possible that those difficult-to-find arbitrage opportunities are insignificant. In such cases, the principle of bounded rationality [14] may offer the following counter argument: there may exist insignificant arbitrage opportunities but it is not worthwhile to exploit it by spending exponential amount of computational resource. On the other hand, the fact that NP-hardness is proven under a set of reasonable assumptions would make one curious in search of truth about computation of arbitrage conditions. In this regard, *complexity on approximability offers an answer to the proposal of bounded rationality in computation*. Here we are interested to determine whether an approximation (within in a factor of $(1+\epsilon)$ for some fixed constant ϵ) to the optimal solution is not possible, or there is a polynomial time approximation scheme (PTAS). Therefore, in order that a quest into the arbitrage condition via computational complexity approach conform to the principle of bounded rationality, we would like to know if we are not able to find a significant large arbitrage opportunities (inapproximability), or more computational time would result in better approximation (PTAS).

The main result of this work deals with the approximability of arbitrage in the foreign exchange market. In Section 2, we introduce the model of an exchange market, together with an NP-hard proof that arbitrage in frictional foreign exchange market. In Section 3, we present our main result of non-approximability. In comparison, we note that two interesting models allow for polynomial time solvability. One is with a central money, much like the role of US dollar set out in the Brenton Woods Agreement. Another is with a market of a constant number of currencies which may be used as a supporting argument for reducing the number of currencies, as is in reality.

2 Models of Exchange Market

Exchange market starts historically in various forms. Barter market, where merchant exchange wheat for eggs, is of the most primitive ones. Foreign exchange market is, in many ways, a virtual barter market, of dollars, marks and yens. Formally speaking, we consider n foreign currencies: $N = \{1, 2, \dots, n\}$. For each directed pair (i, j) , one may change one unit of currency i to r_{ij} units of currency j . Rate r_{ij} is the *exchange rate* from i to j . In an ideal market, the exchange rate holds for any amount that is exchanged. An *arbitrage opportunity* is a set of exchanges between pairs of currencies such that the net balance for each involved currency is non-negative and there is at least one currency for which the net balance is positive. Under ideal market conditions, there is no arbitrage if and only if there is no arbitrage among any three currencies [13].

In reality, various types of *friction* exist. For example, bid-offer spread may be expressed in our mathematical format as $r_{ij} r_{ji} < 1$ for i, j . In addition, usually the traded amount is required to be in multiples of a fixed integer amount, hundreds, thousands or millions. Moreover, different traders may bid or offer at different rates, and each for a limited amount. A more general model to describe these market *imperfections* will include, for pairs $i \neq j \in N$, l_{ij} different rates r_{ij}^k of exchanges from currency i to j up to b_{ij}^k units of currency i , $k = 1, \dots, l_{ij}$, where l_{ij} is the number of different exchange rates from currency i to j .

Let $x = (x_{ij}^k)$ be a currency exchange vector such that

$$(1) \quad 0 \leq x_{ij}^k \leq b_{ij}^k.$$

There exists *arbitrage* in the market if there is a currency exchange vector $x = (x_{ij}^k)$ satisfying the following inequalities with at least one strict inequality:

$$(2) \quad \sum_{j \neq i} \sum_{k=1}^{l_{ji}} [r_{ji}^k x_{ji}^k] - \sum_{j \neq i} \sum_{k=1}^{l_{ij}} x_{ij}^k \geq 0, \quad i = 1, \dots, n.$$

Note that the first term in the right hand side of (2) is the revenue at currency i by selling other currencies and the second term is the expense at currency i by buying other currencies. In addition, we require the currency exchange vector x to be integral, i.e.,

$$(3) \quad x_{ij}^k \text{ is integer, } 1 \leq k \leq l_{ij}, 1 \leq i, j \leq n, i \neq j.$$

Then the problem becomes *NP*-complete.

Theorem 2.1 *It is NP-complete to determine whether there exists arbitrage in a frictional foreign exchange market with bid-ask spreads, bound and integrality constraints.*

Proof. First it is trivial to see the decision problem is in the class *NP*.

Let us now describe a polynomial-time reduction τ from SETCOVER [8,11] to an instance of the decision problem.

Consider an instance \mathcal{I} of SETCOVER: Given a collection $\mathcal{C} = \{C_1, C_2, \dots, C_p\}$ of subsets of $S = \{e_1, e_2, \dots, e_q\}$ and positive integer $K \leq p$, does \mathcal{C} contain a cover for S of size K or less, i.e., a subset $\mathcal{C}' \subseteq \mathcal{C}$ such that every element in S belongs to at least one member of \mathcal{C}' .

We construct an instance $\tau(I)$ of the decision problem with $n = p + q + 4$ from \mathcal{I} in the following way.

- Currencies: $1, 2, \dots, n$.
- Rates:

$$r_{ij} = \begin{cases} 1 & \text{if } 1 \leq i \leq p, p+1 \leq j \leq p+q \text{ and } e_{j-p} \in C_i, \\ 1 & \text{if } p+1 \leq i \leq p+q \text{ and } j = n-3, \\ \frac{1}{q} & \text{if } i = n-3 \text{ and } j = n-2, \\ K+1 & \text{if } i = n-2 \text{ and } j = n-1, \\ 1 & \text{if } i = n-1 \text{ and } j = n, \\ |C_j| & \text{if } i = n \text{ and } 1 \leq j \leq p, \\ 0 & \text{otherwise.} \end{cases}$$

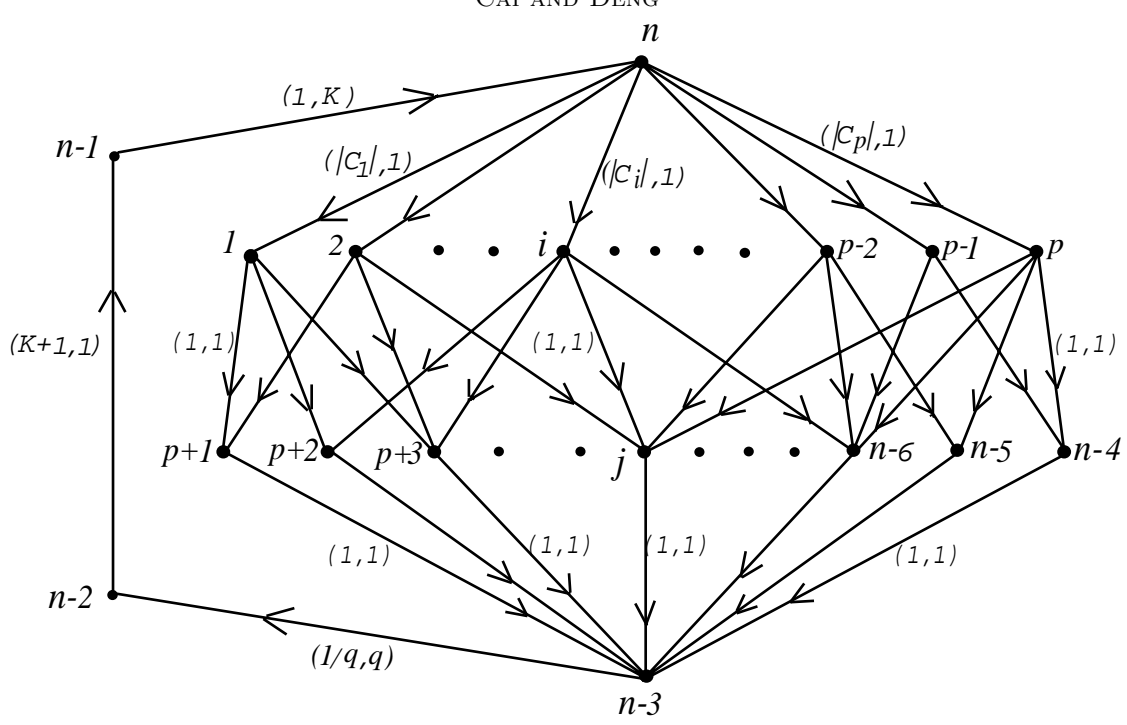
- Bounds:

$$b_{ij} = \begin{cases} q & \text{if } i = n-3 \text{ and } j = n-2, \\ K & \text{if } i = n-2 \text{ and } j = n-1, \\ 1 & \text{if } r_{ij} > 0, ij \neq (n-3)(n-2) \text{ and } ij \neq (n-1)n, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the construction of $\tau(I)$. It is shown by the following graph G_1 .

Now we claim that there exists arbitrage in market $\tau(I)$ if and only if \mathcal{C} contains a cover for S of size K or less.

We first suppose that there exists arbitrage in market $\tau(I)$, i.e., there is a vector $x = (x_{ij})$ satisfying (1), (2) and (3) with at least one strict inequality in (2). Obviously, $x \neq 0$, in particular $x_{(n-2)(n-1)} > 0$. By the integrality of x and the bound $b_{(n-2)(n-1)} = 1$, $x_{(n-2)(n-1)} = 1$, from which it follows easily that $x_{(n-3)(n-2)} = q$, implying $x_{i(n-3)} = 1$ for $i = p+1, \dots, p+q$. We derive further that, for each $i = p+1, \dots, p+q$, there is $h(i)$, $1 \leq h(i) \leq p$, such that $x_{h(i)i} = 1$ and $x_{nh(i)} = 1$, implying $e_{i-p} \in C_{h(i)}$ by the definition of r_{ij} , and hence $\mathcal{C}^* = \{C_j \in \mathcal{C} \mid x_{nj} = 1\}$ is a cover for S . Now let us show $|\mathcal{C}^*| \leq K$.

Fig. 1. Graph G_1

Indeed, it follows from (2) and (3) that

$$|\mathcal{C}^*| = \sum_{j=1}^p x_{nj} \leq \lfloor r_{(n-1)n} x_{(n-1)n} \rfloor \leq K.$$

Conversely suppose that \mathcal{C} contains a cover $\mathcal{C}' = \{C_{i_1}, \dots, C_{i_{K'}}\}$ for S with $K' \leq K$. We define currency exchange $x = (x_{ij})$ as follows.

$$x_{ij} = \begin{cases} 1 & \text{if } C_i \in \mathcal{C}' \text{ and } e_{j-p} \in C_i, \\ 1 & \text{if } p+1 \leq i \leq p+q \text{ and } j = n-3, \\ q & \text{if } i = n-3 \text{ and } j = n-2, \\ 1 & \text{if } i = n-2 \text{ and } j = n-1, \\ K' & \text{if } i = n-1 \text{ and } j = n, \\ 1 & \text{if } i = n \text{ and } C_j \in \mathcal{C}', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $x = (x_{ij})$ satisfies (1)–(3) with strict inequality for $i = n-1$ in (2). Hence there exists arbitrage in market $\tau(I)$. The theorem is proved. \square

3 Non-approximability in General and Two Polynomially Solvable Special Cases

Now we turn to consider the *optimal version* of the arbitrage problem in a frictional foreign exchange market with bid-ask spreads, bound and integrality constraints. It can be formulated as the following integer linear programming (P):

$$\begin{aligned}
 & \text{maximize } \sum_{i=1}^n w_i \left\{ \sum_{j \neq i} \sum_{k=1}^{l_{ji}} \lfloor r_{ji}^k x_{ji}^k \rfloor - \sum_{j \neq i} \sum_{k=1}^{l_{ij}} x_{ij}^k \right\} \\
 (4) \quad (P) : & \text{subject to } \sum_{j \neq i} \sum_{k=1}^{l_{ji}} \lfloor r_{ji}^k x_{ji}^k \rfloor - \sum_{j \neq i} \sum_{k=1}^{l_{ij}} x_{ij}^k \geq 0, \quad i = 1, \dots, n, \\
 (5) \quad & 0 \leq x_{ij}^k \leq b_{ij}^k, \quad 1 \leq k \leq l_{ij}, \quad 1 \leq i, j \leq n, \quad i \neq j, \\
 (6) \quad & x_{ij}^k \text{ integer}, \quad 1 \leq k \leq l_{ij}, \quad 1 \leq i, j \leq n, \quad i \neq j,
 \end{aligned}$$

where $w_i > 0$ is a given weight for currency i , $i = 1, 2, \dots, n$.

Then we have a further inapproximability result.

Theorem 3.1 *There exists fixed $\epsilon > 0$ such that approximating Problem (P) within a factor of n^ϵ is NP-hard.*

Proof. As approximating the MAX-SET-PACKING problem within a factor of n^ϵ is NP-hard for some fixed $\epsilon > 0$ [2,9,15], to prove the theorem, it suffices to show that the MAX-SET-PACKING problem is polynomially reducible to Problem (P).

Consider an instance \mathcal{I} of MAX-SET-PACKING : Given a collection $\mathcal{C} = \{C_1, C_2, \dots, C_p\}$ of subsets of $S = \{e_1, e_2, \dots, e_q\}$, find a subcollection of disjoint subsets $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'|$ is maximized.

We construct an instance $\tau(I)$ of Problem (P) with $n = 2p + q + 1$ from \mathcal{I} in the following way.

- Currencies: $1, 2, \dots, n$.
- Rates:

$$r_{ij} = \begin{cases} \frac{1}{|C_i|} & \text{if } i = n \text{ and } 1 \leq j \leq p, \\ |C_i| + 1 & \text{if } 1 \leq i \leq p \text{ and } j = p + i, \\ 1 & \text{if } p + 1 \leq i \leq p + q, \quad 2p + 1 \leq j \leq 2p + q \text{ and } e_{j-2p} \in C_{i-p}, \\ 1 & \text{if } 2p + 1 \leq i \leq 2p + q \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

- Bounds:

$$b_{ij} = \begin{cases} |C_j| & \text{if } i = n \text{ and } 1 \leq j \leq p, \\ 1 & \text{if } i < n \text{ and } r_{ij} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- weights: $w_i = 1$ for $i = 1, 2, \dots, n$.

This completes the construction of $\tau(I)$. It is shown by the above graph G_2 .

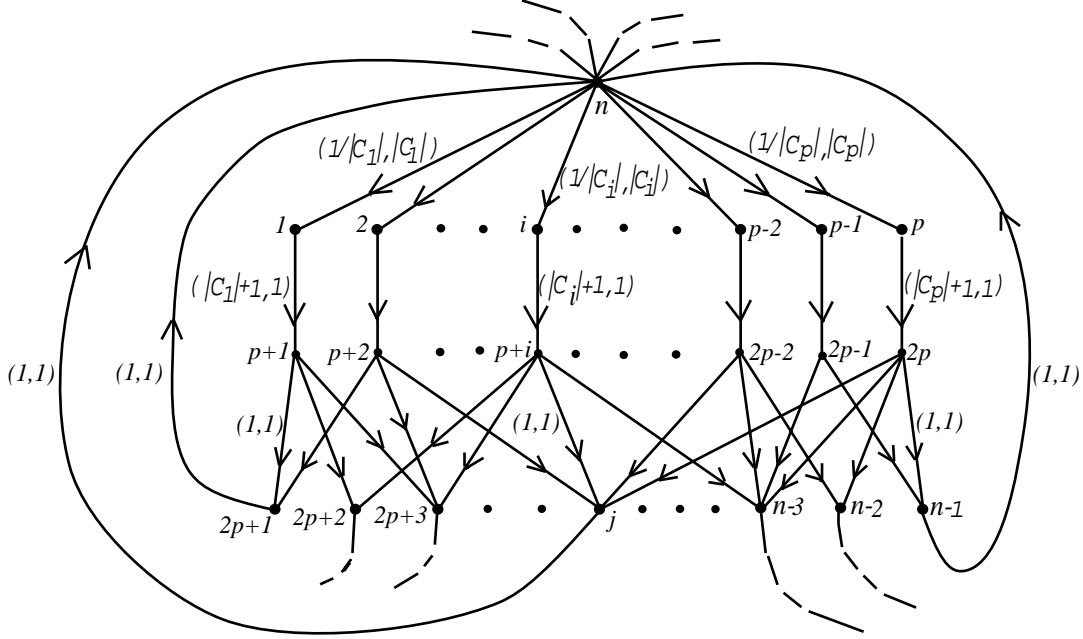


Fig. 2. Graph G_2

Now let us show that each feasible solution of $\tau(I)$ corresponds to a set packing of I with the same objective function value and vice versa.

First, given a feasible solution $x' = (x'_{ij})$ to $\tau(I)$, we define a subcollection

$$\mathcal{C}' = \{C_i \in \mathcal{C} \mid 1 \leq i \leq p \text{ and } x'_{i(i+p)} = 1\}.$$

Then it derives from (4)–(6) that

$$x'_{ni} = |C_i| \quad \text{if } C_i \in \mathcal{C}' \text{ and } 1 \leq i \leq p,$$

$$(7) \quad \sum_{i=2p+1}^{n-1} x'_{in} \geq \sum_{C_i \in \mathcal{C}'} |C_i|,$$

$$(8) \quad \sum \{x'_{ij} \mid 2p+1 \leq j \leq 2p+q, e_{j-2p} \in C_{i-p} \in \mathcal{C}'\} \leq |C_i|, \quad p+1 \leq i \leq 2p,$$

$$(9) \quad \sum \{x'_{ij} \mid 2p+1 \leq j \leq 2p+q, e_{j-2p} \in C_{i-p} \notin \mathcal{C}'\} = 0, \quad p+1 \leq i \leq 2p,$$

$$(10) \quad \sum \{x'_{ij} \mid p+1 \leq i \leq 2p, e_{j-2p} \in C_{i-p}\} \geq x'_{jn}, \quad 2p+1 \leq j \leq n-1.$$

Using (7)–(10) we have

$$\begin{aligned} \sum_{C_i \in \mathcal{C}'} |C_i| &\leq \sum_{i=2p+1}^{n-1} x'_{in} \leq \sum_{j=2p+1}^{n-1} \sum \{x'_{ij} \mid p+1 \leq i \leq 2p, e_{j-2p} \in C_{i-p}\} \\ &= \sum_{i=p+1}^{2p} \sum \{x'_{ij} \mid 2p+1 \leq j \leq 2p+q, e_{j-2p} \in C_{i-p}\} \leq \sum_{C_i \in \mathcal{C}'} |C_i|, \end{aligned}$$

implying all the inequalities in (7), (8) and (10) hold with equality and $x'_{ni} = 0$ if $C_i \notin \mathcal{C}'$. Thus for $1 \leq i \leq p$

$$\begin{aligned} \frac{x'_{ni}}{|C_i|} - x'_{i(i+p)} &= 0, \\ (|C_i| + 1)x'_{i(i+p)} - \sum \{x'_{(i+p)j} \mid e_{j-2p} \in C_i\} &= \begin{cases} 1 & \text{if } C_i \in \mathcal{C}', \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, the objective function value of x' is equal to $|\mathcal{C}'|$.

To prove \mathcal{C}' being a set packing, it suffices to show that the members of \mathcal{C}' are mutually disjoint. Suppose, to the contrary, that there exists $e_j \in S$ belonging to members $C_{i'}$ and $C_{i''}$ of \mathcal{C}' . As all inequalities in (8) and (10) become equalities, we have

$$x'_{(i'+p)(j+2p)} = x'_{(i''+p)(j+2p)} = 1, \quad \sum \{x'_{(i+p)(j+2p)} \mid 1 \leq i \leq p, e_j \in C_i\} = x'_{(j+2p)n},$$

yielding $1 = b_{(j+2p)n} \geq x'_{(j+2p)n} \geq x'_{(i'+p)(j+2p)} + x'_{(i''+p)(j+2p)} = 2$, a contradiction. So \mathcal{C}' is a set packing.

Conversely, for a given set packing $\mathcal{C}' \subseteq \mathcal{C}$, we define a solution $x = (x_{ij})$ of $\tau(I)$ as follows.

$$x_{ij} = \begin{cases} |C_j| & \text{if } i = n, 1 \leq j \leq p \text{ and } C_j \in \mathcal{C}', \\ 1 & \text{if } 1 \leq i \leq p, j = i + p \text{ and } C_i \in \mathcal{C}', \\ 1 & \text{if } p \leq i \leq 2p, 2p+1 \leq j \leq n-1 \text{ and } e_{j-2p} \in C_{i-p} \in \mathcal{C}', \\ 1 & \text{if } 2p+1 \leq i \leq n-1, j = n \text{ and } \mathcal{C}' \text{ covers } e_{i-2p}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that the defined $x = (x_{ij})$ is a feasible solution to $\tau(I)$ and its objective function value is $|\mathcal{C}'|$.

Therefore approximating Problem (P) with a factor of n^ϵ is NP-hard for some fixed $\epsilon > 0$. This completes the proof. \square

However, the maximum value of arbitrage can be found in polynomial time for two special cases when the number of currencies is constant or the exchange graph is star-shaped (a graph is star-shaped if all its edges have a common vertex).

Theorem 3.2 *There are polynomial time algorithms for Problem (P) if the number of currencies is a constant or the exchange graph is star-shaped.*

First, if the exchange graph $G = (V, E)$ is star-shaped, clearly Problem (P) can be polynomially solved since there exist only exchanges between the center currency and the other currencies in the market.

Next we consider the special case of a constant number of currencies. Writing y_{ji}^k for $\lfloor r_{ji}^k x_{ji}^k \rfloor$, Problem (P) can be reformulated as the following integer linear programming (P^*):

$$\begin{aligned}
 & \text{maximize } \sum_{i=1}^n w_i \left\{ \sum_{j \neq i} \sum_{k=1}^{l_{ji}} y_{ji}^k - \sum_{j \neq i} \sum_{k=1}^{l_{ij}} x_{ij}^k \right\} \\
 (P^*) : & \text{subject to } \sum_{j \neq i} \sum_{k=1}^{l_{ji}} y_{ji}^k - \sum_{j \neq i} \sum_{k=1}^{l_{ij}} x_{ij}^k \geq 0, \quad i = 1, \dots, n, \\
 & r_{ji}^k x_{ji}^k \geq y_{ji}^k \quad 1 \leq k \leq l_{ji}, \quad 1 \leq i, j \leq n, \quad i \neq j, \\
 & 0 \leq x_{ij}^k \leq b_{ij}^k, \quad 1 \leq k \leq l_{ij}, \quad 1 \leq i, j \leq n, \quad i \neq j, \\
 & x_{ij}^k, y_{ij}^k \text{ integer}, \quad 1 \leq k \leq l_{ij}, \quad 1 \leq i, j \leq n, \quad i \neq j.
 \end{aligned}$$

Hence (P) can be solved in polynomial time [12].

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